Exercises for Stochastic Processes

Tutorial exercises:

T1. Fix $K > 0$. For $x \in V$, fix $\beta_x, \delta_x \in (0, K]$ and define

$$c(x, \eta) := \beta_x \mathbb{1}_{\{\eta(x)=0\}} + \delta_x \mathbb{1}_{\{\eta(x)=1\}} \,.$$

(a) Show that there exists a spin system $(\eta_t)_{t\geq 0}$ with rate function c.

(b) Prove that it is ergodic and find its invariant distribution.

T2. Let (V, E) be a graph with bounded degree. For $x, y \in V$ we write $x \sim y$ if $\{x, y\} \in E$. Set

$$c(x, \eta) := |\{y \in V \mid y \sim x, \eta(y) = \eta(x)\}| \,.$$

(a) Show that there exists a spin system $(\eta_t)_{t\geq 0}$ with rate function c.

(b) Check its ergodicity for the cases

- $V := \mathbb{Z}^d$, $E := \{\{x, y\} \,:\, \|x - y\|_1 = 1\}$ with $d \in \mathbb{N}$,
- $V := \mathbb{Z}/m\mathbb{Z}$, $E := \{\{x, y\} \,:\, x = y + 1\}$ for m even,
- $V := \mathbb{Z}/m\mathbb{Z}$, $E := \{\{x, y\} \,:\, x = y + 1\}$ for m odd.

T3. Let (η_t) and (ζ_t) be two spin systems satisifying the assumptions of Lemma 5.5. Write down the generator for the coupled process (η_t, ζ_t) for which $\eta_t \leq \zeta_t$ almost surely.

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Homework exercises:

H1. Let p be a stochastic matrix on V with $p(x,x) = 0$ for all $x \in V$. Fix $\beta, \delta \geq 0$ and set

$$c(x,\eta) := \sum_{y:\eta(y)\neq\eta(x)} p(x,y) + \beta \mathbb{1}_{\{\eta(x)=0\}} + \delta \mathbb{1}_{\{\eta(x)=1\}} .$$

(a) Show that there exists a spin system $(\eta_t)_{t\geq 0}$ with rate function c.

(b) Give necessary and sufficient conditions for its ergodicity.

H2. Let $\alpha > 0$. We consider the noisy contact process on $\mathbb{Z}$ given by

$$c(x,\eta) = \begin{cases} 1 & \text{if } \eta(x) = 1, \\ \frac{1}{2}\left(\mathbb{1}_{\{\eta(x-1)=1\}} + \mathbb{1}_{\{\eta(x+1)=1\}}\right) + \alpha & \text{if } \eta(x) = 0. \end{cases}$$

(a) Show that there exists a spin system $(\eta_t)_{t\geq 0}$ with rate function c.

(b) Show that it is ergodic for $\alpha > 0$.

(c) Let π^α be the invariant distribution for the spin system with $\alpha > 0$ and let μ^α be the product measure

$$\mu^\alpha := \prod_{z\in\mathbb{Z}} \mu_z^\alpha$$

with $\mu_z^\alpha(\{0\}) = \frac{1}{1+\alpha}$ and $\mu_z^\alpha(\{1\}) = \frac{\alpha}{1+\alpha}$ for all $z \in \mathbb{Z}$. Show that $\mu^\alpha \preceq \pi^\alpha$ for all $\alpha > 0$.

H3. Consider the voter model on $V := \mathbb{Z}$ given by

$$c(x,\eta) := \frac{1}{2}\left(\mathbb{1}_{\{\eta(x+1)\neq\eta(x)\}} + \mathbb{1}_{\{\eta(x-1)\neq\eta(x)\}}\right) .$$

The initial configuration $\eta_0 \in \{0,1\}^V$ is given by

$$\eta_0(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ 1 & \text{if } x \geq 1 . \end{cases}$$

Show that $\delta_{\eta_0} T_t$ converges weakly as $t \to \infty$ and determine the limit.

Deadline: Monday, 03.02.20